

Free Convection

7.1 INTRODUCTION

Now we shall consider situations for which there is no forced velocity, yet convection currents exist within the fluid. They originate when a body force acts on a fluid in which there are density gradients. The net effect is a buoyancy force, which induces fluid velocity. The density gradient is due to a temperature gradient, and the body force is due to the gravitational field. The seminal contributions of Ostrach [1] and Gebhart, Jaluria et al. [2] provide excellent insight into natural convective flows and its importance in several areas of energy transfer.

7.2 FREE CONVECTION OVER VERTICAL FLAT PLATE

To begin with, the focus is on a classical example of boundary layer development on a heated vertical plate (Fig. 7.1). The fluid close to plate is less dense than fluid that is further removed. Buoyancy forces therefore induce a free convection boundary layer in which the heated fluid rises vertically, entraining fluid from quiescent region. The resulting velocity distribution is unlike that associated with forced convection boundary layers. In particular, the velocity is zero as $y \rightarrow \infty$, as well as at $y = 0$. With one exception, assume the fluid to be incompressible. The exception involves accounting for the effect of density variation due to temperature causing buoyancy force (the Boussinesq approximation). The appropriate form of the x -momentum equation is then

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} - \rho g + \mu \frac{\partial^2 u}{\partial y^2} \quad (7.1)$$

Outside the boundary layer, we can write

$$\frac{\partial p}{\partial x} = -\rho_{\infty} g \quad (7.2)$$

Substituting (7.2) into (7.1), we obtain the following equation

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = (\rho_{\infty} - \rho)g + \mu \frac{\partial^2 u}{\partial y^2} \quad (7.3)$$

Introducing the volume coefficient of expansion $\beta = \frac{1}{\forall} \left(\frac{\partial \forall}{\partial T} \right)_p$, $\forall \equiv$ volume, we can write

$$\beta = \frac{1}{\forall_\infty} \left(\frac{\forall - \forall_\infty}{T - T_\infty} \right) = \frac{\rho_\infty}{1} \left(\frac{\rho_\infty - \rho}{\rho \rho_\infty} \right) \frac{1}{T - T_\infty}$$

or

$$\beta = \frac{(\rho_\infty - \rho)}{\rho(T - T_\infty)} = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p \quad (7.4)$$

From (7.3) and (7.4) we can write

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \beta g \rho (T - T_\infty) + \mu \frac{\partial^2 u}{\partial y^2} \quad (7.5)$$

Similar to forced flow boundary layer, y momentum equation takes the form, $\frac{\partial p}{\partial y} = 0$. The set of equations governing natural convection over a vertically heated plate becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7.6)$$

$$\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = g \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} \quad (7.7)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (7.8)$$

with $\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p = \frac{1}{\rho} \left(\frac{p}{RT^2} \right) = \frac{1}{T}$ (this might be needed for calculating β for perfect gases).

Boundary conditions at $y = 0 : u = 0, v = 0$; at $y = \delta : u = 0, \frac{\partial u}{\partial y} = 0$; at $y = 0 : T = T_w$; at $y = \delta : T = T_\infty$ and $\frac{\partial T}{\partial y} = 0$. Integration of Eq. (7.7) within the boundary layer yields

$$\int_0^\delta \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] dy = \int_0^\delta g \beta (T - T_\infty) dy + \int_0^\delta \nu \frac{\partial^2 u}{\partial y^2} dy$$

Using continuity equation

$$\int_0^\delta \left[\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y} (uv) \right] dy = g \beta \int_0^\delta (T - T_\infty) dy - \nu \frac{\partial u}{\partial y} \Big|_{y=0}$$

or

$$\frac{d}{dx} \int_0^\delta u^2 dy + [uv]_0^\delta = g \beta \int_0^\delta (T - T_\infty) dy - \nu \frac{\partial u}{\partial y} \Big|_{y=0}$$

or

$$\frac{d}{dx} \int_0^\delta u^2 dy = g \beta \int_0^\delta (T - T_\infty) dy - \nu \frac{\partial u}{\partial y} \Big|_{y=0} \quad (7.9)$$

Integration of Eq. (7.8) yields

$$\int_0^\delta \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) dy = \int_0^\delta \alpha \frac{\partial^2 T}{\partial y^2} dy$$

Integrating the second term by parts and invoking from Eq. (7.6), we get

$$\begin{aligned} \int_0^\delta \left(u \frac{\partial T}{\partial x} \right) dy + \int_0^\delta T \frac{\partial u}{\partial x} dy - T_\infty \int_0^\delta \frac{\partial u}{\partial x} dy &= -\alpha \left. \frac{\partial T}{\partial y} \right|_{y=0} \\ \int_0^\delta \frac{\partial}{\partial x} \{ u(T - T_\infty) \} dy &= -\alpha \left. \frac{\partial T}{\partial y} \right|_{y=0} \\ \frac{d}{dx} \int_0^\delta u(T - T_\infty) dy &= -\alpha \left. \frac{\partial T}{\partial y} \right|_{y=0} \end{aligned} \quad (7.10)$$

Defining $\theta = T - T_\infty$ and substituting this in Eq. (7.10), yields

$$\frac{d}{dx} \int_0^\delta u \theta dy = -\alpha \left. \frac{\partial \theta}{\partial y} \right|_{y=0} \quad (7.11)$$

Defining $\theta_m = T_w - T_\infty$, temperature distribution may be assumed as

$$\frac{\theta}{\theta_m} = \frac{T - T_\infty}{T_w - T_\infty} = \left(1 - \frac{y}{\delta} \right)^2 \quad (7.12)$$

This satisfies at $y = 0, T = T_w$ i.e., at $y = 0, \theta = \theta_m$; at $y = \delta, T = T_\infty$ i.e., at $y = \delta, \theta = 0$; at $y = \delta, \frac{\partial T}{\partial y} = 0$ i.e., at $y = \delta, \frac{\partial}{\partial y} \left[(T_w - T_\infty) \left(1 - \frac{y}{\delta} \right)^2 \right] = 0$.

The nature of velocity profile shown in Fig. 7.1 requires velocity in the boundary layer to be of the form

$$u = a + by + cy^2 + dy^3 \quad (7.13)$$

The boundary conditions are at $y = 0, u = 0$; at $y = \delta, u = 0$; at $y = 0, \nu \frac{\partial^2 u}{\partial y^2} = -\beta g (T_w - T_\infty)$; at $y = \delta, \frac{\partial u}{\partial y} = 0$. These boundary conditions will finally produce

$$a = 0, \quad b = \frac{\delta \beta g}{4\nu} (T_w - T_\infty), \quad c = -\frac{\beta g}{2\nu} (T_w - T_\infty), \quad d = \frac{\beta g}{4\nu \delta} (T_w - T_\infty)$$

Substituting these values Eq. (7.13) we get

$$u = \frac{g\beta\delta^2 (T_w - T_\infty)}{4\nu} \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 \quad (7.14)$$

The terms $(T_w - T_\infty)$, δ^2 , $g\beta/4\nu$ can together form the characteristic velocity, u_x so that we can write

$$\frac{u}{u_x} = \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 \quad (7.15)$$

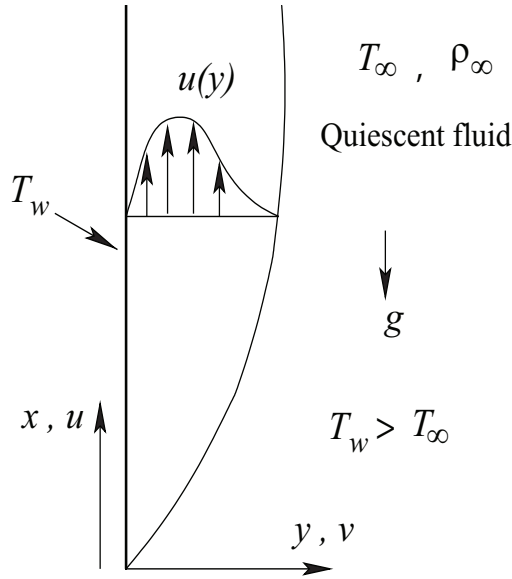


Figure 7.1 Boundary layer development on a heated vertical plate

The maximum velocity and its position (distance in y direction) at any x can be obtained from Eq. (7.15)

$$\frac{d}{dy}(u) = \frac{d}{dy} \left[u_x \left\{ \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 \right\} \right] = 0 \quad (7.16)$$

$$\frac{d}{dy} \left[\frac{y}{\delta} \left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2} \right) \right] = 0$$

$$\text{or } \frac{1}{\delta} - \frac{4y}{\delta^2} + \frac{3y^2}{\delta^3} = 0$$

$$\text{or } \delta^2 - 4\delta y + 3y^2 = 0$$

$$(\delta - y)(\delta - 3y) = 0 \quad (7.17)$$

At $y = \delta$ velocity is zero.

Therefore at $y = \frac{\delta}{3}$, velocity is maximum and

$$\frac{u_{max}}{u_x} = \frac{1}{3} \left[1 - \frac{1}{3} \right]^2$$

$$\text{or, } u_{max} = \frac{4}{27} u_x$$

Energy equation, Eq. (7.11), can be now written as

$$\frac{d}{dx} \int_0^\delta \theta_m \left(1 - \frac{y}{\delta} \right)^2 u_x \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 dy = -\alpha \theta_m \frac{\partial}{\partial y} \left[\left(1 - \frac{y}{\delta} \right)^2 \right] \Big|_{y=0}$$

or

$$\frac{1}{30} \frac{d}{dx} (u_x \delta) \theta_m = \frac{2\alpha \theta_m}{\delta} \quad (7.18)$$

The momentum equation, Eq. (7.9), may be written as

$$\frac{d}{dx} \int_0^\delta \left[u_x \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 \right]^2 dy = g\beta \int_0^\delta \theta_m \left(1 - \frac{y}{\delta} \right)^2 dy - \nu u_x \frac{d}{dy} \left[\frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 \right]_{y=0}$$

or

$$\frac{1}{105} \frac{d}{dx} (u_x^2 \delta) = \frac{g\beta\theta_m\delta}{3} - \frac{\nu u_x}{\delta} \quad (7.19)$$

In Eqs. (7.18) and (7.19), u_x and δ are dependant variables and x is the independant variable. To solve the above two equations, u_x and δ are considered as functions of x .

Assume u_x and δ vary according to the following functions

$$u_x = C_1 x^m \quad \text{and} \quad \delta = C_2 x^n$$

Substitution in Eq. (7.18) will produce

$$\frac{m+n}{30} C_1 C_2 (x)^{m+n-1} = \frac{2\alpha}{C_2} x^{-n} \quad (7.20)$$

Substitution in Eq. (7.19) will produce

$$\frac{2m+n}{105} C_1^2 C_2 (x)^{2m+n-1} = \frac{1}{3} g\beta\theta_m C_2 x^n - \frac{\nu C_1 x^{m-n}}{C_2} \quad (7.21)$$

The equations have to be treated in the following way. First let us consider Eq. (7.20). If a monomial nonlinear expression is dependent on a variable (say x) that is equivalent to an expression independent of x , then for universal validity of the equation as a function of x , the power of the nonlinear expression must vanish. If we divide the entire Eq. (7.20) by its right-hand side, then for the universal validity of the expression, we can say

$$m + 2n - 1 = 0$$

Now, let us look at Eq. (7.21). Any number of terms with non-zero degree can combine to result into a single term, provided each of the terms has same degree. The degree of the resulting single term will be equal to that of other combining terms. This will lead to

$$2m + n - 1 = n \quad \text{and} \quad 2m + n - 1 = m - n$$

or

$$2m - 1 = 0 \quad \text{and} \quad m + 2n - 1 = 0$$

The conclusion derived from Eq. (7.20) and one of the two conclusions derived from Eq. (7.21) is the same. Now by solving these, we get

$$m = 1/2 \quad \text{and} \quad n = 1/4$$

Substituting the values of m and n in Eq. (7.20) and Eq. (7.21), we get

$$C_1 C_2^2 = 80\alpha \quad (7.22)$$

$$\frac{C_1^2 C_2}{84} = \frac{1}{3} g \beta \theta_m C_2 - \frac{C_1}{C_2} \nu \quad (7.23)$$

From Eq. (7.22) we get

$$C_1 = \frac{80\alpha}{C_2^2}$$

Substitution in Eq. (7.23) produces

$$\left(\frac{80\alpha}{C_2^2} \right)^2 \frac{C_2}{84} = \left(\frac{1}{3} g \beta \theta_m \right) C_2 - \frac{80\alpha}{C_2^2} \frac{\nu}{C_2}$$

or

$$76\alpha^2 = \left(\frac{1}{3} g \beta \theta_m \right) C_2^4 - 80\alpha\nu$$

or

$$\left(\frac{1}{3} g \beta \theta_m \right) C_2^4 = 76\alpha^2 + 80\alpha\nu$$

or

$$C_2 = \alpha^{1/4} \left(\frac{76}{80} + \frac{\nu}{\alpha} \right)^{1/4} (80\alpha)^{1/4} \left(\frac{1}{3} \frac{g \beta \theta_m}{\nu^2} \right)^{-1/4} \nu^{-1/2}$$

or

$$C_2 = 3.93 \left(\frac{\nu}{\alpha} \right)^{-1/2} \left(\frac{\nu}{\alpha} + \frac{20}{21} \right)^{1/4} \left(\frac{g \beta \theta_m}{\nu^2} \right)^{-1/4}$$

The back substitution in the expression for C_1 produces

$$C_1 = \frac{80\alpha}{C_2^2} = 5.17 \nu \left(\frac{\nu}{\alpha} + \frac{20}{21} \right)^{-1/2} \left(\frac{g \beta \theta_m}{\nu^2} \right)^{1/2}$$

Therefore, the values of C_1 and C_2 are given as

$$C_1 = 5.17 \nu \left[\frac{\nu}{\alpha} + \frac{20}{21} \right]^{-1/2} \left(\frac{g \beta \theta_m}{\nu^2} \right)^{1/2} \quad (7.24)$$

$$\text{and, } C_2 = 3.93 \left(\frac{\nu}{\alpha} \right)^{-1/2} \left[\frac{\nu}{\alpha} + \frac{20}{21} \right]^{1/4} \left(\frac{g \beta \theta_m}{\nu^2} \right)^{-1/4} \quad (7.25)$$

Substituting C_1 and C_2 for evaluating u_x and δ and substituting Pr for ν/α we get

$$u_x = (5.17) \nu \left(Pr + \frac{20}{21} \right)^{-1/2} \left(\frac{g \beta \theta_m}{\nu^2} \right)^{1/2} x^{1/2}$$

$$\delta = 3.93 (Pr)^{-1/2} \left(Pr + \frac{20}{21} \right)^{1/4} \left(\frac{g \beta \theta_m}{\nu^2} \right)^{-1/4} x^{1/4}$$